

# AFFINE HALL-LITTLEWOOD FUNCTIONS FOR $A_1^{(1)}$ AND SOME CONSTANT TERM IDENTITIES OF CHEREDNIK-MACDONALD-MEHTA TYPE

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**ABSTRACT.** We study  $t$ -analogs of string functions for integrable highest weight representations of the affine Kac-Moody algebra  $A_1^{(1)}$ . We obtain closed form formulas for certain  $t$ -string functions of levels 2 and 4. As corollaries, we obtain explicit identities for the corresponding affine Hall-Littlewood functions, as well as higher-level generalizations of Cherednik's Macdonald and Macdonald-Mehta constant term identities.

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be an untwisted affine Kac-Moody algebra. Let  $\delta$  denote its null root,  $P^+$  denote the set of dominant integral weights and let  $t$  be an indeterminate. To each  $\lambda \in P^+$ , one can associate a Hall-Littlewood function  $P_\lambda(t)$ ; these are affine generalizations of the classical Hall-Littlewood polynomials [12] and interpolate between the Weyl-Kac characters  $\chi_\lambda$  and the Weyl group orbit sums  $m_\lambda$  [6, 17]. The affine Kostka-Foulkes polynomials  $K_{\lambda\mu}(t)$  are defined to be the coefficients that appear in the relation  $\chi_\lambda = \sum_{\mu \in P^+} K_{\lambda\mu}(t) P_\mu(t)$ . Affine Hall-Littlewood functions have been studied in various contexts by several authors [3, 4, 6, 10, 17]. Affine Kostka-Foulkes polynomials are also of great interest, especially on account of their positivity properties [4].

One has a well developed theory of integrable highest weight representations of affine Kac-Moody algebras. Given  $\lambda \in P^+$ , let  $L(\lambda)$  denote the corresponding highest weight representation. The generating functions for weight multiplicities along  $\delta$ -strings in  $L(\lambda)$  gives (upto a multiplicative factor) the Kac-Moody string functions [9, (12.7.7)]. Since affine Hall-Littlewood functions  $P_\lambda(t)$  are  $t$ -deformations of the  $m_\lambda$ , the coefficients  $K_{\lambda\mu}(t)$  above may be viewed as  $t$ -analogs of weight multiplicities. The notion of a string function thus admits a natural  $t$ -analog; these so called  $t$ -string functions [6, 17] are defined to be generating functions for the  $K_{\lambda\mu}(t)$  along  $\delta$ -strings.

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It is known (see [3] for an extensive account) that affine Hall-Littlewood functions are closely related to the theory of the *Double Affine Hecke algebra* (DAHA). In fact, our notion is a special case (corresponding to dominant affine weights) of the more general notion of affine *Hall functions of level*  $l > 0$  introduced in [3]. The level 1 case has been studied for all affine root systems [2, 3, 15]; for simply laced  $\mathfrak{g}$  and  $\lambda$  of level 1, there are closed form formulas for  $P_\lambda(t)$  and the corresponding  $t$ -string function. In fact, the explicit formula for the level 1  $t$ -string function is equivalent to Cherednik's Macdonald-Mehta constant term identity [2, 17]. However, there is almost nothing known about higher level affine Hall-Littlewood functions.

In this article, we consider the rank 1 affine Kac-Moody algebra  $A_1^{(1)}$ . We study higher level  $t$ -string functions with a view toward obtaining explicit formulas. We exhibit such formulas for certain level 2 and 4  $t$ -string functions; in turn these determine the corresponding affine Hall-Littlewood functions. Interpreted in terms of the *constant term* functional [2], our results provide new constant term identities generalizing those of Macdonald and Macdonald-Mehta (in the  $A_1^{(1)}$  case).

Our method is based on studying *principal specializations* of affine Hall-Littlewood functions. When  $\mathfrak{g} = A_1^{(1)}$ , these can be related to certain bilateral  $q$ -hypergeometric series. For our special dominant weights, the resulting series can be summed in closed form using various specializations of Bailey's classical  ${}_6\psi_6$  summation.

Our methods are very specific to rank one and we do not know if they can be extended to general untwisted affines, or even to  $A_n^{(1)}$ . Nevertheless, our results provide the first explicitly computable examples of higher level affine Hall-Littlewood functions and we hope they will be of value in studying the role of the DAHA in this theory.

The article is arranged as follows. In §2, we recall the main aspects of the theory of Hall-Littlewood functions associated to affine (and more generally symmetrizable Kac-Moody) Lie algebras. In §3, we work with  $A_1^{(1)}$ , and compute principal specializations of Hall-Littlewood functions using classical summation formulas for our special cases. Section 4 contains our main results; specifically theorems 2-4 compute closed form expressions for the  $t$ -string functions and corollaries 1-5 give formulas for the affine Hall-Littlewood functions and the new constant term identities. The short appendix collects together classical summation theorems for  $q$ -hypergeometric series that are used in this article.

## 2. HALL-LITTLEWOOD FUNCTIONS

2.1. We first recall relevant facts and notation concerning Hall-Littlewood functions associated to symmetrizable Kac-Moody algebras [4, 6, 17]. Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra with Cartan subalgebra  $\mathfrak{h}$ . Let  $\alpha_i, i = 1 \cdots n$  be the simple roots of  $\mathfrak{g}$  and  $P, Q, P^+, Q^+$  be the weight lattice, the root lattice and the sets of dominant weights and non-negative

integer linear combinations of simple roots respectively. We will denote the Weyl group of  $\mathfrak{g}$  by  $W$  and let  $(\cdot, \cdot)$  be a nondegenerate,  $W$ -invariant symmetric bilinear form on  $\mathfrak{h}^*$ . Let  $\ell(\cdot)$  be the length function on  $W$  and for  $\lambda \in \mathfrak{h}^*$ , let  $W_\lambda \subset W$  denote the stabilizer of  $\lambda$ . Let  $\Delta$  (resp.  $\Delta_+$ ,  $\Delta_-$ ) be the set of roots (resp. positive, negative roots) of  $\mathfrak{g}$ . We will also let  $\Delta^{re}$  (resp.  $\Delta_\pm^{re}$ ) denote the set of real roots of  $\mathfrak{g}$  (resp. positive/negative real roots), and similarly  $\Delta^{im}$  (resp.  $\Delta_\pm^{im}$ ), the corresponding subsets of imaginary roots. Given  $\lambda, \mu \in \mathfrak{h}^*$ , we say  $\lambda \geq \mu$  if  $\lambda - \mu \in Q^+$ . Given  $\lambda \in \mathfrak{h}^*$ , define  $D(\lambda) := \{\gamma \in \mathfrak{h}^* | \gamma \leq \lambda\}$ . Let  $\mathcal{E}_t$  be the set of all series of the form

$$(2.1) \quad \sum_{\lambda \in \mathfrak{h}^*} c_\lambda(t) e^\lambda$$

where each  $c_\lambda(t) \in \mathbb{C}[[t]]$  and  $c_\lambda = 0$  outside the union of a finite number of sets of the form  $D(\mu)$ ,  $\mu \in \mathfrak{h}^*$ . For each  $\alpha \in \Delta$ , let  $\text{mult}(\alpha)$  denote the root multiplicity of  $\alpha$ . Let  $\rho$  be a Weyl vector of  $\mathfrak{g}$  defined by  $(\rho, \alpha_i^\vee) = 1 \ \forall i = 1 \cdots n$ , where as usual  $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$ . Finally, given  $\lambda \in P^+$ , let  $L(\lambda)$  be the integrable  $\mathfrak{g}$ -module with highest weight  $\lambda$ .

For  $\lambda \in P^+$ , the *Hall-Littlewood function*  $P_\lambda(t)$  is defined as

$$(2.2) \quad P_\lambda(t) := \frac{1}{W_\lambda(t)} \frac{\sum_{w \in W} (-1)^{\ell(w)} w \left( e^{\lambda + \rho} \prod_{\alpha \in \Delta_+} (1 - te^{-\alpha})^{\text{mult}(\alpha)} \right)}{e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}$$

where  $W_\lambda(t) = \sum_{\sigma \in W_\lambda} t^{\ell(\sigma)}$  is the Poincaré series of  $W_\lambda$ . Proposition 1 of [17] implies that  $P_\lambda(t)$  is a well defined element of  $\mathcal{E}_t$ . If  $\chi_\lambda$  denotes the formal character of the irreducible highest weight representation  $L(\lambda)$  of  $\mathfrak{g}$ , we can write

$$(2.3) \quad \chi_\lambda = \sum_{\mu \in P^+, \mu \leq \lambda} K_{\lambda\mu}(t) P_\mu(t).$$

By corollary 1 of [17], the *Kostka-Foulkes polynomials*  $K_{\lambda\mu}(t)$  lie in  $\mathbb{Z}[t]$ . One also has  $P_\lambda(0) = \chi_\lambda$  and  $P_\lambda(1) = m_\lambda := \sum_{\mu \in W_\lambda} e^\mu$ . As a consequence,  $K_{\lambda\mu}(0) = \delta_{\lambda,\mu}$  and  $K_{\lambda\mu}(1) = \dim(L(\lambda)_\mu)$ .

Define the special element  $\tilde{\Delta} \in \mathcal{E}_t$  by

$$\tilde{\Delta} := \frac{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}{\prod_{\alpha \in \Delta_+} (1 - te^{-\alpha})^{\text{mult}(\alpha)}} = \prod_{\alpha \in \Delta_+} [(1 - e^{-\alpha})(1 + te^{-\alpha} + t^2e^{-2\alpha} + \dots)]^{\text{mult}(\alpha)}$$

Letting  $S(w) := \{\alpha \in \Delta_+ : w^{-1}\alpha \in \Delta_-\}$ , we have the following useful relation (equation (4.2) of [17]).

$$(2.4) \quad e^{-\lambda} \tilde{\Delta} P_\lambda(t) = \frac{1}{W_\lambda(t)} \sum_{w \in W} e^{w\lambda - \lambda} \prod_{\alpha \in S(w)} \frac{t - e^{-\alpha}}{1 - te^{-\alpha}}$$

2.2. Specializing to the case that  $\mathfrak{g}$  is an untwisted affine Kac-Moody algebra, we let  $\overset{\circ}{\mathfrak{g}}$  denote the underlying finite dimensional simple Lie algebra

of rank  $l$ , say. If  $\delta$  is the null root, then  $\Delta_+^{im} = \{k\delta : k \geq 1\}$  and each imaginary root has multiplicity  $l$ .

Suppose  $\lambda \in P^+$ , let  $\text{Max}(\lambda) := \{\mu \in P^+ : \mu \leq \lambda; \mu + \delta \not\leq \lambda\}$ . For each  $\mu \in \text{Max}(\lambda)$ , the generating function

$$(2.5) \quad a_\mu^\lambda(t, q) := \sum_{k \geq 0} K_{\lambda, \mu - k\delta}(t) q^k$$

is termed a *t-string function*. The *constant term* map  $\text{ct}(\cdot)$  [11] is defined on elements  $f = \sum_\lambda f_\lambda e^\lambda$  of  $\mathcal{E}_t$  by

$$\text{ct}(f) := \sum_{k \in \mathbb{Z}} f_{k\delta} e^{k\delta}.$$

If  $g \in \mathcal{E}_t$  is such that  $g = \text{ct}(g)$ , then  $\text{ct}(fg) = g \text{ct}(f)$  for all  $f \in \mathcal{E}_t$  [11, (3.1)].

We will let  $q := e^{-\delta}$  for the rest of the paper. The key relationship between the notions of the preceding paragraph is the following [17, (5.8)]:

$$(2.6) \quad a_\mu^\lambda(t, q) = \text{ct}(e^{-\mu} \tilde{\Delta} \chi_\lambda)$$

### 3. $A_1^{(1)}$ AND BILATERAL $q$ -HYPERGEOMETRIC SERIES

For the rest of the article we will restrict ourselves to the simplest affine Kac-Moody algebra  $A_1^{(1)}$ . The goal of this section is to compute the principal specializations of certain Hall-Littlewood functions associated to  $A_1^{(1)}$ . Let  $\alpha_0, \alpha_1$  denote the simple roots of  $A_1^{(1)}$ ; the null root  $\delta = \alpha_0 + \alpha_1$ . The real roots  $\{\alpha_i + k\delta : k \in \mathbb{Z}, i = 0, 1\}$  and imaginary roots  $\{k\delta : k \in \mathbb{Z} \setminus \{0\}\}$  all have multiplicity 1. We will assume that the form  $(\cdot, \cdot)$  is normalized such that  $(\alpha_i, \alpha_i) = 2$  for  $i = 0, 1$ . The Weyl group  $W$  is the infinite dihedral group generated by the simple reflections  $r_0, r_1$ ; we write  $W := \{w_j : j \in \mathbb{Z}\}$  where  $w_j := (r_0 r_1)^{\frac{j}{2}}$  for  $j$  even and  $w_j := r_0 (r_1 r_0)^{\frac{j-1}{2}}$  for  $j$  odd. It is easily seen that  $S(w_0) = \emptyset$  and  $S(w_j)$  equals  $\{\alpha_0, \alpha_0 + \delta, \dots, \alpha_0 + (j-1)\delta\}$  for  $j > 0$  and  $\{\alpha_1, \alpha_1 + \delta, \dots, \alpha_1 + (|j|-1)\delta\}$  for  $j < 0$ . For all  $j \in \mathbb{Z}$  one has  $\rho - w_j \rho = \sum_{\gamma \in S(w_j)} \gamma = j\alpha_0 + \binom{j}{2}\delta$ .

Let  $\pi(w_j) := \prod_{\alpha \in S(w_j)} \frac{t - e^{-\alpha}}{1 - te^{-\alpha}}$ . One observes [11, (4.2)] that

$$\pi(w_j) = t^{|j|} \prod_{\alpha \in S(w_j)} \frac{1 - t^{-1}e^{-\alpha}}{1 - te^{-\alpha}} = t^j \frac{(t^{-1}e^{-\alpha_0}; e^{-\delta})_j}{(te^{-\alpha_0}; e^{-\delta})_j}$$

for all  $j \in \mathbb{Z}$ . We have used the usual  $q$ -hypergeometric notations:  $(a; w)_\infty := \prod_{i=0}^\infty (1 - aw^i)$  and  $(a; w)_j := (a; w)_\infty / (aw^j; w)_\infty$  for all  $j \in \mathbb{Z}$ . We will also use the shorthand  $(a_1, a_2, \dots, a_k; w)_j := (a_1; w)_j (a_2; w)_j \cdots (a_k; w)_j$ .

Next, for  $\gamma = c_0\alpha_0 + c_1\alpha_1 \in Q$ , let  $\text{ht}(\gamma) := c_0 + c_1 = (\gamma, \rho)$ . Given  $f := \sum_{\beta \in Q^+} f_\beta e^{-\beta}$  with  $f_\beta \in \mathbb{C}[[t]]$ , the principal specialization

$$\mathbb{F}(f) := \sum_{\beta \in Q^+} f_\beta v^{\text{ht}(\beta)} = \sum_{\beta \in Q^+} f_\beta v^{(\beta, \rho)} \in \mathbb{C}[[t, v]].$$

For instance,

$$(3.1) \quad \mathbb{F}(\tilde{\Delta}) = \frac{(v, v, v^2; v^2)_\infty}{(tv, tv, tv^2; v^2)_\infty} \text{ and } \mathbb{F}(\pi(w_j)) = t^j \frac{(t^{-1}v; v^2)_j}{(tv; v^2)_j}.$$

Since  $\mathbb{F}(e^{-\alpha_0}) = \mathbb{F}(e^{-\alpha_1})$ , we have  $\mathbb{F}(\pi(w_j)) = \mathbb{F}(\pi(w_{-j}))$ . Further  $\mathbb{F}(e^{w_j\lambda-\lambda}) = v^{(\lambda-w_j\lambda, \rho)} = v^{(\lambda, \rho-w_j^{-1}\rho)}$ . Observe also that  $w_j^{-1}$  is either  $w_j$  or  $w_{-j}$  depending on whether  $j$  is odd or even. The above remarks together with equation (2.4) imply that

$$\mathbb{F}(W_\lambda(t)e^{-\lambda}\tilde{\Delta}P_\lambda(t)) = \sum_{j \in \mathbb{Z}} v^{(\lambda, \rho-w_j\rho)} \mathbb{F}(\pi(w_j)).$$

Let  $\lambda$  be of level  $l \geq 0$ , i.e.,  $(\lambda, \delta) = l$ . If  $(\lambda, \alpha_0) = p$ , one has  $0 \leq p \leq l$  and  $(\lambda, \rho - w_j\rho) = (\lambda, j\alpha_0 + \binom{j}{2}\delta) = pj + l\binom{j}{2}$ . Let  $F_{l,p}(t, v) := \mathbb{F}(W_\lambda(t)e^{-\lambda}\tilde{\Delta}P_\lambda(t))$ .

Using the principal specialization of  $\pi(w_j)$ , we obtain

$$(3.2) \quad F_{l,p}(t, v) = \sum_{j \in \mathbb{Z}} v^{pj+l\binom{j}{2}} t^j \frac{(t^{-1}v; v^2)_j}{(tv; v^2)_j} = \sum_{j \in \mathbb{Z}} \left[ \frac{1 + v^{(l-2p)j}}{2} \right] v^{pj+l\binom{j}{2}} t^j \frac{(t^{-1}v; v^2)_j}{(tv; v^2)_j}$$

where for the last equality, we used the fact that  $\mathbb{F}(\pi(w_j)) = \mathbb{F}(\pi(w_{-j}))$ .

If a closed form expression for  $F_{l,p}(t, v)$  can be found, equations (3.1) and (3.2) allow us to determine the principal specialization of  $P_\lambda(t)$  via the relation

$$(3.3) \quad \mathbb{F}(e^{-\lambda}P_\lambda(t)) = \frac{(tv, tv, tv^2; v^2)_\infty}{(v, v, v^2; v^2)_\infty} \frac{F_{l,p}(t, v)}{W_\lambda(t)}.$$

We now show that the sum defining  $F_{l,p}(t, v)$  can be written as an explicit infinite product for certain special values of  $(l, p)$ . These are

$$(l, p) = (0, 0), (1, 0), (1, 1), (2, 0), (2, 1), (2, 2), (4, 1), (4, 3).$$

We use classical summation theorems for bilateral  $q$ -hypergeometric series for this purpose (see Appendix). Note that the symmetry between  $\alpha_0$  and  $\alpha_1$  ensures that  $F_{l,p}(t, v) = F_{l, l-p}(t, v)$ ; it is thus enough to consider  $0 \leq p \leq [l/2]$ . When  $(l, p)$  equals  $(0, 0)$  or  $(1, 0)$ , we recover the identities of Macdonald [11] and Fishel-Grojnowski-Teleman [4, 6] respectively.

3.1.  $l = 0, p = 0$ : In this case, as shown by Macdonald [11],  $F_{0,0}(t, v)$  is just the  ${}_1\psi_1$  bilateral  $q$ -hypergeometric series. Taking  $a = t^{-1}v, b = tv, w = v^2, z = t$  in equation (5.2) of the Appendix, and using equation (3.3), one

obtains (cf [11, equation (4.3)]):

$$(3.4) \quad \mathbb{F}(e^{-\lambda} P_{\lambda}(t)) = \frac{(t^2 v^2; v^2)_{\infty}}{(tv^2; v^2)_{\infty}}$$

3.2.  $l = 1, p = 0$ :

$$F_{1,0}(t, v) = \sum_{j \in \mathbb{Z}} v^{(j)} \frac{t^j (t^{-1} v; v^2)_j}{(tv; v^2)_j} \left[ \frac{1 + v^j}{2} \right] = \sum_{j \in \mathbb{Z}} v^{(j)} t^j \frac{(\sqrt{t^{-1} v}, -\sqrt{t^{-1} v}; v)_j}{(\sqrt{tv}, -\sqrt{tv}; v)_j} \left[ \frac{1 + v^j}{2} \right].$$

This can be summed by specializing Bailey's  ${}_6\psi_6$  identity (equation (5.3)) at  $w = v, b = \sqrt{t^{-1} v}, c = -\sqrt{t^{-1} v}, d = a^{\frac{1}{2}}$  and letting  $e \rightarrow \infty, a \rightarrow 1$ . The resulting identity is

$$(3.5) \quad \mathbb{F}(e^{-\lambda} P_{\lambda}(t)) = \frac{(t^2 v^2; v^2)_{\infty}}{(v; v^2)_{\infty}}$$

The unspecialized version of this identity is implicit in [4, 6, 17].

3.3.  $l = 2, p = 1$ : In this case, we first set  $u = v^2$ .

$$F_{2,1}(t, v) = \sum_{j \in \mathbb{Z}} u^{(j)} (tu^{1/2})^j \frac{(t^{-1} u^{1/2}; u)_j}{(tu^{1/2}; u)_j}.$$

We recognize this as the specialization of the  ${}_6\psi_6$  series at  $w = u, b = t^{-1} u^{1/2}, c = a^{\frac{1}{2}}, d = -c$  and  $e \rightarrow \infty, a \rightarrow 1$ . We thereby obtain

$$(3.6) \quad \mathbb{F}(e^{-\lambda} P_{\lambda}(t)) = \frac{(tv^2; v^2)_{\infty} (t^2 v^2; v^4)_{\infty}}{(v, v, -v^2; v^2)_{\infty}}.$$

3.4.  $l = 2, p = 0$ : Again, with  $u = v^2$ , we have

$$F_{2,0}(t, v) = \sum_{j \in \mathbb{Z}} u^{(j)} \frac{1 + u^j}{2} \frac{t^j (t^{-1} u^{1/2}; u)_j}{(tu^{1/2}; u)_j}$$

We now apply Bailey's  ${}_6\psi_6$  with  $w = u, b = t^{-1} u^{1/2}, c = -a^{\frac{1}{2}} u^{1/2}, d = a^{\frac{1}{2}}$  and let  $e \rightarrow \infty, a \rightarrow 1$  to get:

$$(3.7) \quad \mathbb{F}(e^{-\lambda} P_{\lambda}(t)) = \frac{(tv; v)_{\infty} (-tv^2; v^2)_{\infty}}{(v, v, -v; v^2)_{\infty}}$$

3.5.  $l = 4, p = 1$ : Setting  $u = v^2$  as above gives

$$F_{4,1}(t, v) = \sum_{j \in \mathbb{Z}} u^{2(j)} (tu^{1/2})^j \frac{(t^{-1} u^{1/2}; u)_j}{(tu^{1/2}; u)_j} \left[ \frac{1 + u^j}{2} \right].$$

This is again a specialization of the  ${}_6\psi_6$  identity, this time at  $w = u, b = t^{-1} u^{1/2}, c = a^{\frac{1}{2}}$  and  $e \rightarrow \infty, a \rightarrow 1$ . Thus:

$$(3.8) \quad \mathbb{F}(e^{-\lambda} P_{\lambda}(t)) = \frac{(tv; v)_{\infty}}{(v, v; v^2)_{\infty}}$$

**Remark 1:** (a). Besides the already known cases of levels 0 and 1, we have thus obtained infinite product expressions for  $\mathbb{F}(e^{-\lambda}P_\lambda(t))$  for all dominant weights of level 2 and some of level 4. A natural question is whether these exhaust the cases where such infinite product expressions exist. It is easy to see from equation (3.2) that for any dominant weight  $\lambda$ , further specializing  $t = v^g$  for an odd natural number  $g$ , reduces  $F_{l,p}(t, v)$  to a rational function of  $v$ . When  $F_{l,p}(t, v)$  admits an infinite product expression involving terms of the form  $(\pm t^i v^j; v^k)_\infty$  as above, all zeros and poles of these rational functions have modulus 1. Computational data (using MAPLE) and this latter observation suggest that the only  $\lambda$ 's of level  $\leq 10$  which admit such infinite product expressions are the ones we have already found.

(b). We also recall that at  $t = 0$  and  $t = 1$ ,  $e^{-\lambda}P_\lambda(t)$  reduces to  $e^{-\lambda}\chi_\lambda$  and  $e^{-\lambda}m_\lambda$  respectively. The principal specializations of both these have well known infinite product expressions *for all*  $\lambda \in P^+$  (by [9, (10.10.1)] and the Jacobi triple product identity respectively). Given  $\lambda \in P^+$  with  $(\lambda, \delta) = l > 0$ ,  $(\lambda, \alpha_0) = p$ , one has

$$(3.9) \quad \mathbb{F}(e^{-\lambda}\chi_\lambda) = \frac{(v^{p+1}, v^{l-p+1}, v^{l+2}; v^{l+2})_\infty}{(v, v, v^2; v^2)_\infty}$$

$$(3.10) \quad \mathbb{F}(e^{-\lambda}m_\lambda) = \frac{1}{\#W_\lambda}(-v^p, -v^{l-p}, v^l; v^l)_\infty$$

For our special  $\lambda$ 's, equations (3.5)-(3.8) interpolate between these two expressions (the elementary identity  $(-w; w)_\infty = 1/(w; w^2)_\infty$  is useful in checking this explicitly).

#### 4. MAIN THEOREMS

We now use the principal specializations of §3 to determine the  $t$ -string function  $a_\mu^\lambda(t, q)$  for the special  $\lambda$ 's of level 2 and 4 and all  $\mu \in \text{Max}(\lambda)$ . For ease of notation, let  $\Lambda_0, \Lambda_1$  denote (any choice of) fundamental weights, i.e., satisfying  $(\Lambda_i, \alpha_j) = \delta_{ij}$  for  $i, j = 0, 1$ . We recall that explicit expressions for the level 0 and 1  $t$ -string functions  $a_0^0(t, q)$  and  $a_{\Lambda_0}^{\Lambda_0}(t, q)$  are known [4, 6, 11, 17]:

$$a_0^0(t, v) = \frac{(tq; q)_\infty}{(t^2q; q)_\infty} \quad \text{and} \quad a_{\Lambda_0}^{\Lambda_0}(t, v) = \frac{1}{(t^2q; q)_\infty}.$$

These are respectively equivalent to the Macdonald and Macdonald-Mehta constant term identities for  $A_1^{(1)}$ . In Theorem 1 below, we state these constant term identities. But first some notation: let

$$\hat{\mu} := \prod_{\alpha \in \Delta_+^{re}} \frac{1 - e^{-\alpha}}{1 - te^{-\alpha}} = \frac{(e^{-\alpha_1}, qe^{\alpha_1}; q)_\infty}{(te^{-\alpha_1}, tqe^{\alpha_1}; q)_\infty} \quad (\text{the Cherednik kernel})$$

and let  $\Theta_1(e^{-\alpha_0}, e^{-\alpha_1}) = \sum_{j \in \mathbb{Z}} (e^{-\alpha_0})^{j^2} (e^{-\alpha_1})^{j^2-j} = \sum_{j \in \mathbb{Z}} e^{j\alpha_1} q^{j^2}$ . We recognize  $\Theta_1 = \Theta_1(e^{-\alpha_0}, e^{-\alpha_1})$  as the theta function of the root lattice of the underlying finite dimensional simple Lie algebra  $A_1 \cong sl_2$ .

**Theorem 1.** (Cherednik) *For the affine Lie algebra  $A_1^{(1)}$ , one has*

$$(1) \text{ ct}(\hat{\mu}) = \frac{(tq; q)_\infty^2}{(t^2q; q)_\infty (q; q)_\infty}$$

$$(2) \text{ ct}(\hat{\mu} \Theta_1) = \frac{(tq; q)_\infty}{(t^2q; q)_\infty} \quad \square$$

For later use, we also define  $\tilde{\Delta}^{im} := \prod_{k \geq 1} \left( \frac{1-e^{-k\delta}}{1-te^{-k\delta}} \right) = \frac{(q; q)_\infty}{(tq; q)_\infty}$ . Observe that

$$(4.1) \quad \tilde{\Delta} = \hat{\mu} \tilde{\Delta}^{im} \text{ and } \text{ct}(\tilde{\Delta}^{im}) = \tilde{\Delta}^{im}.$$

In the next three subsections, we derive closed-form expressions for our  $t$ -string functions of levels 2 and 4. As mentioned in the introduction, these in turn determine the corresponding affine Hall-Littlewood functions and give us new constant term identities of Cherednik-Macdonald-Mehta type.

4.1.  $\lambda = \Lambda_0 + \Lambda_1$ . Then  $\lambda$  is of level 2 and  $\text{Max}(\lambda) = \{\lambda\}$ . Thus

$$(4.2) \quad \chi_\lambda = a_\lambda^\lambda(t, q) P_\lambda(t).$$

Applying the principal specialization, we get  $\mathbb{F}(e^{-\lambda} \chi_\lambda) = a_\lambda^\lambda(t, v^2) \mathbb{F}(e^{-\lambda} P_\lambda(t))$ .

By equation (3.9),  $\mathbb{F}(e^{-\lambda} \chi_\lambda) = \frac{(v^2; v^4)_\infty}{(v, v; v^2)_\infty}$ . Together with equation (3.6), this gives us the following.

**Theorem 2.**

$$a_{\Lambda_0 + \Lambda_1}^{\Lambda_0 + \Lambda_1}(t, q) = \frac{1}{(tq; q)_\infty (t^2q; q^2)_\infty} \quad \square$$

To obtain the corresponding constant term identity, first observe that  $a_\lambda^\lambda(t, q) = \text{ct}(e^{-\lambda} \tilde{\Delta} \chi_\lambda) = a_\lambda^\lambda(1, q) \tilde{\Delta}^{im} \text{ct}(\hat{\mu} e^{-\lambda} m_\lambda)$ . The first equality is by equation (2.6), and the second follows from equation (4.1) and equation (4.2) at  $t = 1$ . For  $\lambda = \Lambda_0 + \Lambda_1$ , it is straightforward to see that

$$e^{-\lambda} m_\lambda = \sum_{\mu \in W \cdot \lambda} e^{\mu - \lambda} = \sum_{j \in \mathbb{Z}} (e^{-\alpha_0})^{\frac{j(j+1)}{2}} (e^{-\alpha_1})^{\frac{j(j-1)}{2}} = \Theta_R(e^{-\alpha_0}, e^{-\alpha_1})$$

where  $\Theta_R$  is the Ramanujan Theta function [1, p. 34]. Observe that  $\Theta_R$  is also equal to  $\sum_{j \in \mathbb{Z}} q^{\binom{j}{2}} (e^{-\alpha_1})^j$ . Theorem 2 thus implies the following level 2 analog of Cherednik's difference Macdonald-Mehta identity, for  $A_1^{(1)}$ .

**Corollary 1.**

$$\text{ct}(\hat{\mu} \Theta_R) = \frac{(q; q^2)_\infty}{(t^2q; q^2)_\infty} \quad \square$$



Next, recalling the well known fact that  $\chi_{\Lambda_0+\Lambda_1} = e^{\Lambda_0+\Lambda_1} \prod_{\alpha \in \Delta_+} (1+e^{-\alpha})$  for  $A_1^{(1)}$  [9, Ex 10.1], one observes that equation (4.2) and theorem 2 together determine the affine Hall-Littlewood function  $P_{\Lambda_0+\Lambda_1}(t)$ :

**Corollary 2.**

$$P_{\Lambda_0+\Lambda_1}(t) = e^{\Lambda_0+\Lambda_1} (-e^{-\alpha_1}, -qe^{\alpha_1}, -q, tq; q)_{\infty} (t^2q; q^2)_{\infty}$$

□

4.2.  $\lambda = 2\Lambda_0$ . Again,  $\lambda$  is of level 2, with  $\text{Max}(\lambda) = \{2\Lambda_0, 2\Lambda_0 - \alpha_0\}$ . Thus

$$(4.3) \quad \chi_{2\Lambda_0} = a_{2\Lambda_0}^{2\Lambda_0}(t, q) P_{2\Lambda_0}(t) + a_{2\Lambda_0-\alpha_0}^{2\Lambda_0}(t, q) P_{2\Lambda_0-\alpha_0}(t)$$

We observe that  $(2\Lambda_0 - \alpha_0, \alpha_0) = 0$  and  $(2\Lambda_0 - \alpha_0, \alpha_1) = 2$ . The  $\alpha_0 \leftrightarrow \alpha_1$  symmetry implies  $\mathbb{F}(e^{-2\Lambda_0} P_{2\Lambda_0}(t)) = \mathbb{F}(e^{-(2\Lambda_0-\alpha_0)} P_{2\Lambda_0-\alpha_0}(t))$ . Principally specializing equation (4.3), one obtains

$$(4.4) \quad \mathbb{F}(e^{-2\Lambda_0} \chi_{2\Lambda_0}) = (a_{2\Lambda_0}^{2\Lambda_0}(t, v^2) + v a_{2\Lambda_0-\alpha_0}^{2\Lambda_0}(t, v^2)) \mathbb{F}(e^{-2\Lambda_0} P_{2\Lambda_0}(t)).$$

We now set  $v \mapsto -v$ , and use equations (3.7), (3.9) to determine the individual  $t$ -string functions. The result is summarized in the following theorem.

For ease of notation, we let  $[a]_{\infty} := (a; q)_{\infty}$  in the rest of the article.

**Theorem 3.** (1)  $a_{2\Lambda_0}^{2\Lambda_0}(t, q) = \frac{[-tq^{\frac{1}{2}}]_{\infty} + [tq^{\frac{1}{2}}]_{\infty}}{2[t^2q]_{\infty}}$

(2)  $a_{2\Lambda_0-\alpha_0}^{2\Lambda_0}(t, q) = \frac{[-tq^{\frac{1}{2}}]_{\infty} - [tq^{\frac{1}{2}}]_{\infty}}{2q^{\frac{1}{2}} [t^2q]_{\infty}}$

□

This theorem allows us to derive two further constant term identities. To state these, we introduce  $\Theta_2 = \Theta_2(e^{-\alpha_0}, e^{-\alpha_1}) := \sum_{j \in \mathbb{Z}} (e^{-\alpha_0})^{2j^2} (e^{-\alpha_1})^{2j^2-2j}$ . Observing that  $\Theta_2 = e^{-2\Lambda_0} m_{2\Lambda_0}$ , it is clear that  $\Theta_2(e^{-\alpha_0}, e^{-\alpha_1}) = \Theta_1(e^{-2\alpha_0}, e^{-2\alpha_1})$ . We also let  $\hat{\Theta}_2 := \Theta_2(e^{-\alpha_1}, e^{-\alpha_0})$ . With this notation, we claim the following.

**Corollary 3.**

(1)  $\text{ct}(\hat{\mu} \Theta_2) = \frac{q}{1-q} \frac{[tq]_{\infty}}{[t^2q]_{\infty}} \left( \frac{[-tq^{\frac{1}{2}}]_{\infty}^{\text{odd}} q^{-\frac{1}{2}} - [-tq^{\frac{1}{2}}]_{\infty}^{\text{even}}}{[-q^{\frac{1}{2}}]_{\infty}^{\text{even}}} \right)$

(2)  $\text{ct}(\hat{\mu} \Theta_2 e^{-\alpha_1}) = \frac{1}{1-q} \frac{[tq]_{\infty}}{[t^2q]_{\infty}} \left( \frac{[-tq^{\frac{1}{2}}]_{\infty}^{\text{even}} - q^{\frac{1}{2}} [-tq^{\frac{1}{2}}]_{\infty}^{\text{odd}}}{[-q^{\frac{1}{2}}]_{\infty}^{\text{odd}} q^{-\frac{1}{2}}} \right)$

where for a series  $\xi = \sum_{k=0}^{\infty} a_k q^{k/2}$ , we let  $\xi^{\text{odd}} := \sum_{k \text{ odd}} a_k q^{k/2}$

and  $\xi^{\text{even}} := \sum_{k \text{ even}} a_k q^{k/2}$ .

*Proof:* Setting  $t = 1$  in equation (4.3) gives:

$$(4.5) \quad \chi_{2\Lambda_0} = a_{2\Lambda_0}^{2\Lambda_0}(1, q) m_{2\Lambda_0} + a_{2\Lambda_0 - \alpha_0}^{2\Lambda_0}(1, q) m_{2\Lambda_0 - \alpha_0}.$$

Again, since  $(2\Lambda_0 - \alpha_0, \alpha_0) = 0$  and  $(2\Lambda_0 - \alpha_0, \alpha_1) = 2$ , the  $\alpha_0 \leftrightarrow \alpha_1$  symmetry argument used before shows  $e^{-(2\Lambda_0 - \alpha_0)} m_{2\Lambda_0 - \alpha_0} = \widehat{\Theta}_2$ . Equations (4.5) and (2.6) thus imply the following relations.

$$(4.6) \quad a_{2\Lambda_0}^{2\Lambda_0}(t, q) = a_{2\Lambda_0}^{2\Lambda_0}(1, q) \text{ct}(\tilde{\Delta}\Theta_2) + a_{2\Lambda_0 - \alpha_0}^{2\Lambda_0}(1, q) \text{ct}(\tilde{\Delta}e^{-\alpha_0}\widehat{\Theta}_2)$$

$$(4.7) \quad a_{2\Lambda_0 - \alpha_0}^{2\Lambda_0}(t, q) = a_{2\Lambda_0}^{2\Lambda_0}(1, q) q^{-1} \text{ct}(\tilde{\Delta}e^{-\alpha_1}\Theta_2) + a_{2\Lambda_0 - \alpha_0}^{2\Lambda_0}(1, q) \text{ct}(\tilde{\Delta}\widehat{\Theta}_2)$$

For  $\beta = c_0\alpha_0 + c_1\alpha_1 \in Q$ , let  $\widehat{\beta} := c_0\alpha_1 + c_1\alpha_0$ . Given  $\xi = \sum_{\beta \in Q^+} f_{\beta} e^{-\beta}$ , define  $\widehat{\xi} := \sum_{\beta \in Q^+} f_{\beta} e^{-\widehat{\beta}}$ . We observe the following easy properties of the hat operation: (i)  $\text{ct}(\xi) = \text{ct}(\widehat{\xi})$ , (ii)  $\widehat{\xi\eta} = \widehat{\xi}\widehat{\eta}$  and (iii)  $\tilde{\Delta}$  is invariant under the hat operation. These imply that  $\text{ct}(\tilde{\Delta}\Theta_2) = \text{ct}(\tilde{\Delta}\widehat{\Theta}_2)$  and  $\text{ct}(\tilde{\Delta}e^{-\alpha_0}\widehat{\Theta}_2) = \text{ct}(\tilde{\Delta}e^{-\alpha_1}\Theta_2)$ . So, equations (4.6) and (4.7) above form a  $2 \times 2$  system :

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = A \begin{bmatrix} \text{ct}(\tilde{\Delta}\Theta_2) \\ \text{ct}(\tilde{\Delta}e^{-\alpha_1}\Theta_2) \end{bmatrix}$$

where  $\zeta_1 = a_{2\Lambda_0}^{2\Lambda_0}(t, q)$ ,  $\zeta_2 = a_{2\Lambda_0 - \alpha_0}^{2\Lambda_0}(t, q)$  and the matrix  $A$  is given by

$$A = \begin{bmatrix} a_{2\Lambda_0}^{2\Lambda_0}(1, q) & a_{2\Lambda_0 - \alpha_0}^{2\Lambda_0}(1, q) \\ q^{-1}a_{2\Lambda_0}^{2\Lambda_0}(1, q) & a_{2\Lambda_0 - \alpha_0}^{2\Lambda_0}(1, q) \end{bmatrix}.$$

Inverting  $A$ , and using equation (4.1) completes the proof of corollary 3.  $\square$

As before, theorem 3 can be used to derive an explicit expression for  $P_{2\Lambda_0}(t)$ . One now uses equation (4.3), and the  $\alpha_0 \leftrightarrow \alpha_1$  symmetry. We content ourselves with stating the result of this calculation.

**Corollary 4.** *With notation as above, one has:*

$$P_{2\Lambda_0}(t) = e^{2\Lambda_0} (\alpha \Theta_2 + \beta e^{-\alpha_0} \widehat{\Theta}_2)$$

with  $\alpha = \frac{1}{2} \left( \frac{[-q^{\frac{1}{2}}]_{\infty}}{[-tq^{\frac{1}{2}}]_{\infty}} + \frac{[q^{\frac{1}{2}}]_{\infty}}{[tq^{\frac{1}{2}}]_{\infty}} \right) \frac{[t^2q]_{\infty}}{[q]_{\infty}}$  and  $\beta = \frac{1}{2q^{\frac{1}{2}}} \left( \frac{[-q^{\frac{1}{2}}]_{\infty}}{[-tq^{\frac{1}{2}}]_{\infty}} - \frac{[q^{\frac{1}{2}}]_{\infty}}{[tq^{\frac{1}{2}}]_{\infty}} \right) \frac{[t^2q]_{\infty}}{[q]_{\infty}}$ .  $\square$

4.3.  $\lambda = 3\Lambda_0 + \Lambda_1$ . In this case,  $\lambda$  is of level 4, with  $\text{Max}(\lambda) = \{3\Lambda_0 + \Lambda_1, 3\Lambda_0 + \Lambda_1 - \alpha_0\}$ . We let  $\Theta_4 := e^{-(3\Lambda_0 + \Lambda_1)} m_{3\Lambda_0 + \Lambda_1}$ . Reasoning as in the previous subsection and using the principal specialization from equation (3.8), one obtains the following theorem and its corollary.

**Theorem 4.** (1)  $a_{3\Lambda_0 + \Lambda_1}^{3\Lambda_0 + \Lambda_1}(t, q) = [-tq]_{\infty} a_{2\Lambda_0}^{2\Lambda_0}(t, q)$ .

$$(2) \ a_{3\Lambda_0+\Lambda_1-\alpha_0}^{3\Lambda_0+\Lambda_1}(t, q) = [-tq]_\infty a_{2\Lambda_0-\alpha_0}^{2\Lambda_0}(t, q). \quad \square$$

**Corollary 5.** (1)  $\text{ct}(\hat{\mu} \Theta_4) = \frac{[-tq]_\infty}{[-q]_\infty} \text{ct}(\hat{\mu} \Theta_2).$

(2)  $\text{ct}(\hat{\mu} \Theta_4 e^{-\alpha_1}) = \frac{[-tq]_\infty}{[-q]_\infty} \text{ct}(\hat{\mu} \Theta_2 e^{-\alpha_1}).$

□

As in level 2, there is an explicit formula for  $P_{3\Lambda_0+\Lambda_1}(t)$  as well; the details are omitted.

**Remark 2:** We observe that the  ${}_6\psi_6$  identity was central to all our  $A_1^{(1)}$   $t$ -string function computations. A generalization of our approach to the higher rank affines  $\mathfrak{g} = A_n^{(1)}$  might be possible by using a suitable multivariable generalization of the  ${}_6\psi_6$  sum. Several such (distinct) generalizations are known [7, 8, 13, 14, 16], and it is an interesting question whether one of these choices leads to explicit formulas for affine Hall-Littlewood functions of small level, for  $A_n^{(1)}$ .

## 5. APPENDIX

For quicker reference and to fix notation, we give below the classical summation formulas for bilateral basic hypergeometric series that have been used in this article.

For parameters  $a_i, b_j$ , recall that the series  ${}_m\psi_n$  is defined by

$$(5.1) \quad {}_m\psi_n \left( \begin{matrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_n \end{matrix}; w, z \right) := \sum_{j \in \mathbb{Z}} \frac{(a_1, a_2, \dots, a_m; w)_j}{(b_1, b_2, \dots, b_n; w)_j} z^j$$

Ramanujan's  ${}_1\psi_1$  and Bailey's  ${}_6\psi_6$  sums [5, Appendix II] are given below.

$$(5.2) \quad {}_1\psi_1 \left( \begin{matrix} a \\ b \end{matrix}; w, z \right) = \frac{(w, \frac{b}{a}, az, \frac{w}{az}; w)_\infty}{(b, \frac{w}{a}, z, \frac{b}{az}; w)_\infty}$$

$$(5.3) \quad {}_6\psi_6 \left( \begin{matrix} wa^{\frac{1}{2}} & -wa^{\frac{1}{2}} & b & c & d & e \\ a^{\frac{1}{2}} & -a^{\frac{1}{2}} & \frac{aw}{b} & \frac{aw}{c} & \frac{aw}{d} & \frac{aw}{e} \end{matrix}; w, \frac{wa^2}{bcde} \right) = \frac{(aw, \frac{aw}{bc}, \frac{aw}{bd}, \frac{aw}{cd}, \frac{aw}{be}, \frac{aw}{ce}, \frac{aw}{de}, w, \frac{w}{a}; w)_\infty}{(\frac{aw}{b}, \frac{aw}{c}, \frac{aw}{d}, \frac{aw}{e}, \frac{w}{b}, \frac{w}{c}, \frac{w}{d}, \frac{w}{e}, \frac{wa^2}{bcde}; w)_\infty}$$

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